Effects of distant boundaries on pattern forming instabilities

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The distinction between convective and absolute instability for an instability in the form of a traveling wave becomes important when the transition takes place in a finite domain. When the instability takes the form of waves traveling in one direction only, or the boundaries are perfectly absorbing, then the absolute stability criterion is the correct one for global modes to exist. When waves can travel equally easily in either direction, on the other hand, the convective criterion is the correct one. In this paper we show how the two results are linked by investigating a model system in which both left and right traveling waves can be unstable, but in which there is no symmetry. Whether the absolute instability boundary or some other boundary is the appropriate one depends on the degree of asymmetry. Both linear and nonlinear aspects of the problem are discussed. [S1063-651X(98)01501-3]

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I. INTRODUCTION

Many theoretical analyses of pattern-forming instabilities of a uniform background make the assumption that the instability takes place in a domain of infinite lateral extent. Then the linearized perturbation equations describing the instability are separable in the lateral variables, and the analysis is thereby much simplified. This approach only makes sense, of course, if the limits of the instability properties in a very large domain tend smoothly to those for an infinite domain. In many cases such an assumption is well justified; for example, Rayleigh-Benard convection in a large box of lateral extent L can be shown to appear at a value of the temperature difference that differs from the value in the infinite layer by an amount $O(L^{-2})$. The convection problem is particularly simple, as it is an example of a steady-state bifurcation. When the instability is of Hopf or oscillatory type, such as the onset of convection in binary mixtures, there are two possible further complications: first, there is a nonlinear selection effect in an infinite layer, which may favor either traveling or standing waves at onset; and second, there is a possible distinction between convective instability, in which a traveling wave pulse, with growing energy, is transient at any particular location, and absolute instability, in which there is growth at all points. It seems likely that both these effects will depend on lateral boundaries, even if they are distant. Cross and co-workers [1–3] carried out extensive analyses of model problems describing the onset of binary fluid convection in a finite container, to see what can occur. They found that, while there is an interesting variety of nonlinear effects, the linear onset of motion occurs generically at a value of the instability parameter (e.g., vertical temperature gradient) that is the same in the limit of large L as in the infinite case.

The binary-fluid problem discussed above has the feature that waves traveling to left and right can be excited with equal ease, and indeed are related by the obvious reflection symmetry even at finite amplitude. There are pattern-forming instabilities, however, that do not possess this symmetry. Of particular interest is the bifurcation leading to dynamo waves in the solar convection zone. Here the unstable waves travel in one direction only. The evolution of these waves has been intensively studied in a finite geometry [7,5], and it is found that while the onset of waves occurs simultaneously in an infinite or a semi-infinite layer, when the layer is of finite extent the onset of instability is delayed by an amount of order unity, even when the size of the layer tends to infinity. This is quite different from the binary fluid problem, and leads to the interesting question of how the transition between the two situations takes place. A natural way to investigate the problem is to utilize the coupled Landau-Ginzburg equations of Cross [1], simplifying by assuming real coefficients (qualitatively similar effects occur when the coefficients are complex). However, we shall generalize the equations' symmetry by removing the left-right symmetry that links the coefficients in the two equations. Thus we write

$$A_T = \mu_1 A + c_1 A_X + \lambda_1 A_{XX} - A^3 - \nu_1 B^2 A, \qquad (1)$$

$$B_T = \mu_2 B - c_2 B_X + \lambda_2 B_{XX} - B^3 - \nu_2 A^2 B, \qquad (2)$$

where $A(X,T)$, $B(X,T)$ are the amplitudes of the wave envelopes. c_i are the group velocities of the packets, assumed positive, so that the two types of wave propagate in opposite directions. The coefficients of A^3 and B^3 have been scaled to be -1 (supposing both to be negative). The equations are to be solved in $0 \leqslant X \leqslant L$, with the boundary conditions $A(0)=0$, $A(L)=-r_2 B'(L)$, $B(0)=r_1 A'(0)$, and $B(L)=0$. This simple model is useful, as it illustrates well the transition from convective to absolute instability (see, e.g., Ref. [6]). It suffices to demonstrate the effects of asymmetry by letting $\mu_1 \neq \mu_2$ and $c_1 \neq c_2$, and so in general we shall take $\lambda_1 = \lambda_2$, $\nu_1 = \nu_2$, and $r_1 = r_2$, and drop the subscripts.

We analyze the system first as a linear eigenvalue problem, and then by conducting numerical simulations in the nonlinear regime. Our focus throughout will be to identify the circumstances in which the boundary conditions at $X=0, L$ remain important even as $L \to \infty$.

II. LINEAR ANALYSIS

We now drop the nonlinear terms in Eqs. (1) and (2), and for convenience write $\mu_1 = \mu - 2\delta$ and $\mu_2 = \mu$, where $\delta \geqslant 0$.

Then we may seek steady solutions for A and B in the forms

$$A=\alpha e^{c_1 X/2\lambda}\sin p_1 X, \quad B=\beta e^{c_2(L-X)/2\lambda}\sin p_2(L-X), \tag{3}$$

where

$$p_1^2=(\mu-2\delta)/\lambda-(c_1/2\lambda)^2 \quad \text{and} \quad p_2^2=\mu/\lambda-(c_2/2\lambda)^2. \tag{4}$$

These functions satisfy half of the boundary conditions. The other conditions give the dispersion relation

$$\sin(p_1 L)\sin(p_2 L)=r^2 p_1 p_2 e^{(c_1+c_2)L/2\lambda}. \tag{5}$$

When $r=0$ the equations are decoupled, and then the critical value of μ coincides with the onset of absolute instability for each of the separate waves. (This situation was thoroughly discussed by Tobias, Proctor, and Knobloch [5,6].) However, it can be seen from the last equation that the case $r=0$ is singular in the limit $L\rightarrow\infty$. When $r\neq 0$, the way μ approaches the limit depends on the size of δ, through the signs of p_1^2 and p_2^2 for large L.

Suppose first that $p_1^2, p_2^2<0$, so let $q_1^2=-p_1^2$ and $q_2^2=-p_2^2$. The dispersion relation becomes

$$\sinh(q_1 L)\sinh(q_2 L)=r^2 q_1 q_2 e^{(c_1+c_2)L/2\lambda}. \tag{6}$$

Now, as $L\rightarrow\infty$ $\sinh(q_1 L)\rightarrow\frac{1}{2}e^{q_1 L}$, and so

$$\begin{aligned} q_1 L+q_2 L &\approx \ln(4r^2)+\ln(q_1 q_2)+(c_1+c_2)L/2\lambda \\ &\approx (c_1+c_2)L/2\lambda \quad \text{as } L\rightarrow\infty. \end{aligned} \tag{7}$$

This equation, together with Eq. (5), can be used to show that

$$q_1=\tfrac{1}{2}[c_1/\lambda+4\delta/(c_1+c_2)], \tag{8}$$

$$q_2=\tfrac{1}{2}[c_2/\lambda-4\delta/(c_1+c_2)] \tag{9}$$

as long as $\delta<(c_1+c_2)c_2/4\lambda\equiv\delta_c$.

So if $\delta<\delta_c$, then the critical value of μ (where the bifurcation takes place) (μ_c) as $L\rightarrow\infty$ is $\mu_c=2\delta c_2/(c_1+c_2)-4\lambda\delta^2/(c_1+c_2)^2+O(L)$, using Eqs. (9) and (4). This is a system where the two waves are strongly interacting, resulting in the bifurcation taking place at a value of μ below the absolute instability threshold for an infinite layer. If $\delta=0$, the bifurcation point is identical with the onset of convective instability, though the eigensolutions are not extended waves but wall modes. This is the scenario explored by Cross and Kuo [3] and Dangelmayr, Knobloch, and Wegelin [4].

When $\delta>\delta_c$, the bifurcation changes its character, since one or the other of p_1^2 or p_2^2 must now be positive (no balance is possible when both are positive in the limit of large L). There are two possibilities, depending on the relative magnitudes of c_1 and c_2. If $p_2^2<0$ and $p_1^2>0$ then we must have $\delta<(c_2^2-c_1^2)/8\lambda$, but this is impossible since $\delta>\delta_c$. Thus we must have $p_1^2<0$ and $p_2^2>0$, and hence

$$\sinh(q_1 L)\sin(p_2 L)=r^2 q_1 p_2 e^{(c_1+c_2)L/2\lambda} \tag{10}$$

so

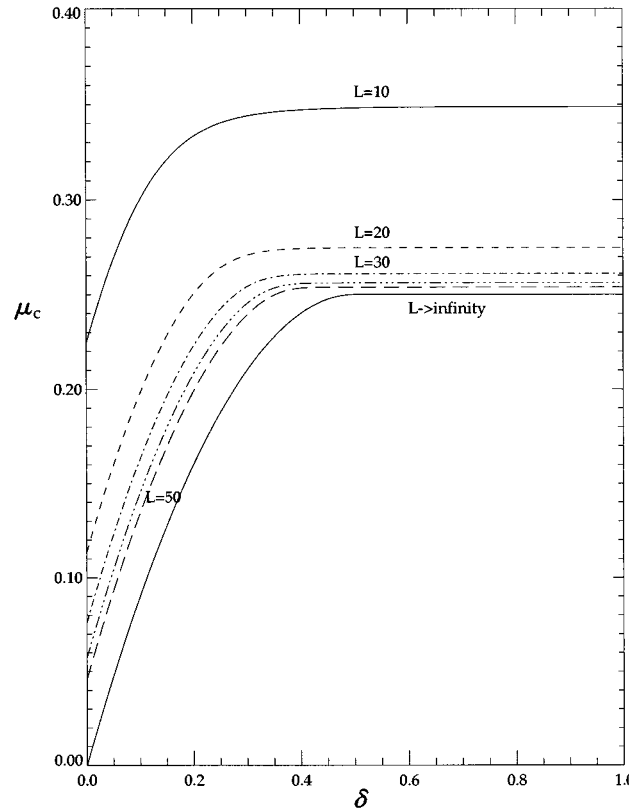

FIG. 1. μ_c vs δ for $L=10$, 20, 30, 40, and 50 with $c_1=c_2=\lambda=1$.

$$\begin{aligned} \ln[\sin(p_2 L)]+\ln[\sinh(q_1 L)] &= \ln(r^2)+\ln(q_1 p_2) \\ &\quad +(c_1+c_2)L/2\lambda. \end{aligned} \tag{11}$$

Thus, as $L\rightarrow\infty$,

$$(c_1+c_2)L/2\lambda\approx q_1 L+\ln[\sin(p_2 L)]. \tag{12}$$

So if we suppose that $e^{[(c_1+c_2)/2\lambda-q_1]L}\ll 1$, i.e., $q_1>(c_1+c_2)/2\lambda$, then

$$p_2\approx\pi/L \quad \text{and so} \quad \mu=\lambda(c_2/2\lambda)^2+\lambda(\pi/L)^2. \tag{13}$$

It can be seen that $q_1>(c_1+c_2)/2\lambda$ corresponds to $\delta>\delta_c+\lambda/2(\pi/L)^2$ which is what we assumed, if L is sufficiently large. Since $\sinh(q_i L)=\pm\sinh(q_i L\pm in\pi)$ there are many nearby complex eigenvalues (and associated eigenmodes) with imaginary part $O(L^{-1})$ and real part $O(L^{-2})$, when $L\gg 1$. Though this does not affect the formal bifurcation structure at finite L, it suggests that weakly nonlinear theory (based on one unstable mode) will be inapplicable except very close to the bifurcation point (see Ref. [6]). Note also that the limits attained in both cases are independent of r, providing that $r\neq 0$.

Thus for $\delta>\delta_c$, $\mu_c=\lambda(c_2/2\lambda)^2+\lambda(\pi/L)^2$ as $L\rightarrow\infty$. This is the same value as if there were no coupling between the two envelopes; the bifurcation occurs at the absolute instability point as $L\rightarrow\infty$, so in this case the large L limit and the semi-infinite case behave differently. This is explained in Ref. [7]. The numerical experiments support this claim; in Fig. 1, $c_1=c_2=\lambda=1$, and the change in behavior therefore takes place at $\delta=\frac{1}{2}$, as is seen.

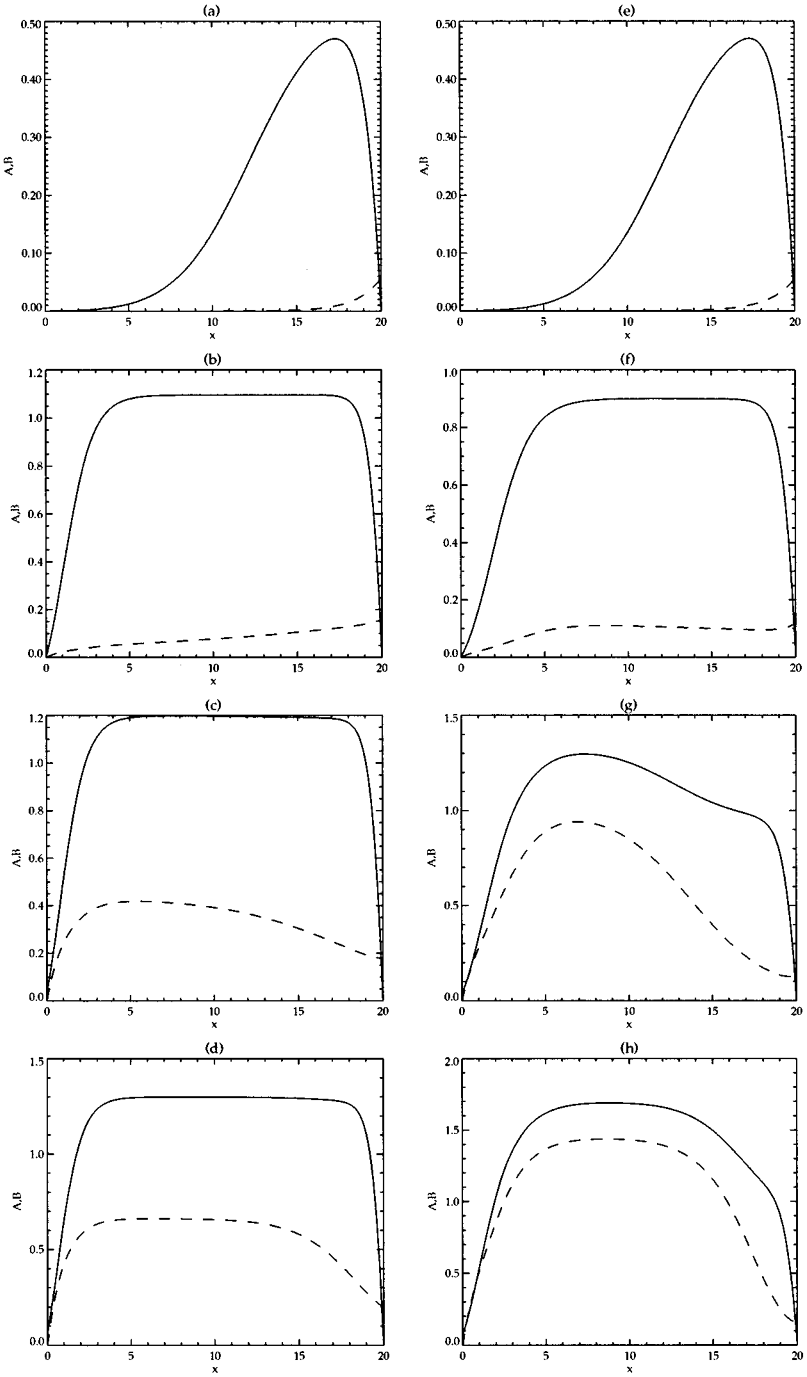

FIG. 2. Graphs of nonlinear steady solutions for A (dashed lines) and B (full lines) for $L=20$, $\delta=0.75$, $\nu=-0.2$ (a)–(d), $\nu=-0.9$ (e)–(h), and (a) $\mu=0.3$, (b) $\mu=1.2$, (c) $\mu=1.4$, (d) $\mu=1.6$, (e) $\mu=0.3$, (f) $\mu=0.8$, (g) $\mu=0.9$, and (h) $\mu=1.0$.

III. NONLINEAR SIMULATIONS

We now restore the nonlinear terms in Eqs. (1) and (2), setting $\nu_1=\nu_2=\nu$ for simplicity. We also simplify by taking $c_1=c_2=1$ and $\lambda=1$. Thus we solve

$$A_T=(\mu-2\delta)A+A_X+A_{XX}-A^3-\nu B^2A, \qquad (14)$$

$$B_T=\mu B-B_X+B_{XX}-B^3-\nu A^2B, \qquad (15)$$

where now we seek to understand the effects of the nonlinear coupling represented by ν in the asymmetric case. Clearly the selection of a pattern is the outcome of a competition between the linear and nonlinear coupling terms. It may be seen, when μ is sufficiently large, that the solutions in the interior of the box are almost independent of X. Pattern selection in that case may be expected to be similar to that for the corresponding ordinary differential equation system

$$A_T=(\mu-2\delta)A-A^3-\nu B^2A, \qquad (16)$$

$$B_T=\mu B-B^3-\nu A^2B. \qquad (17)$$

This equation is in standard form, and been analyzed extensively by many authors. The main conclusions are that, when $|\nu|$ is small, so that the nonlinear coupling is weak, then for sufficiently large μ mixed solutions are stable [though if $\delta>0$ B-modes ($A=0$) are stable for sufficiently small μ]. If ν is sufficiently large, then the single mode solutions are stable. This case has been investigated for $\delta=0$ in Refs. [1,3], so we do not discuss it here. Of more interest for our purposes is the situation for which the value of δ exceeds δ_c, but the nonlinear coupling favors mixed modes. In this case B bifurcates first; then, when $\mu-\mu_c$ is of order unity, B fills the box, and A can be of order r due to the boundary coupling terms. Only when $\mu-2\delta$ exceeds $\frac{1}{4}+\nu B^2$ can a large amplitude solution for A appear, independent of r. In the intermediate convective type regime the solution for A can be of order unity, but the spatial extent of the region scales with $|\ln r|$. Plots of A and B for representative values of the parameters are shown in Fig. 2.

IV. DISCUSSION

This short paper has investigated the conditions under which weak boundary coupling at distant boundaries can influence the onset and form of instabilities of traveling wave type. In it, we have clarified the connection between the situation in which the instability can only travel in one direction, in which case a global mode can exist only above the absolute instability boundary, and the fully symmetric case for which any coupling through the boundaries leads to the convective instability criterion, which is the correct one for long boxes. In the fully nonlinear regime the effects of differences in the growth rate diminish, and the solutions resemble the symmetric case.

ACKNOWLEDGMENTS

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